数理解析研究所講究録
1492 巻 2006 年 78-83

GROUPS OF MEASURE-PRESERVING HOMEOMORPHISMS OF NONCOMPACT 2-MANIFOLDS

京都工芸繊維大学 矢ケ崎 達彦 (TATSUHIKO YAGASAKI)
KYOTO INSTITUTE OF TECHNOLOGY

1. Introduction

A. Fathi [5] made a comprehensive study on topological and algebraic properties of groups of measure-preserving homeomorphisms of compact n-manifolds. R. Belranga [1, 2, 3] has extended this work to the noncompact manifolds. In this article we combine R. Belranga's results with our works on groups of homeomorphisms of noncompact 2-manifolds and obtain some conclusions on topological properties of groups of measure-preserving homeomorphisms of noncompact 2-manifolds [12].

2. Preliminaries on spaces of Radon measures

First we recall some basic facts on spaces of Radon measures and actions of homeomorphism groups. Suppose X is a connected locally connected locally compact separable metrizable space.

2.1. Spaces of Radon measures.

Let $\mathcal{B}(X)$ denote the σ-algebra of Borel subsets of X and $\mathcal{K}(X)$ denote the set of all compact subsets of X

Definition 2.1.

(1) A Radon measure on X is a measure μ on $(X, \mathcal{B}(X))$ such that $\mu(K) < \infty$ for any $K \in \mathcal{K}(X)$.

(2) $\mathcal{M}(X) =$ the space of Radon measures on X.

(3) The weak topology w on $\mathcal{M}(X)$ is the weakest topology such that

$$\Phi_f : \mathcal{M}(X) \to \mathbb{R} : \ \Phi_f(\mu) = \int_M f\, d\mu \quad \text{is continuous}$$

for any continuous function $f : M \to \mathbb{R}$ with compact support.

(4) $\mathcal{M}^A(X) = \{\mu \in \mathcal{M}(X) \mid \mu(A) = 0\}$ $(A \in \mathcal{B}(X))$.

Remark 2.1. Let $K \in \mathcal{K}(X)$.

(1) The map $\mathcal{M}(X)_w \ni \mu \longmapsto \mu(K) \in \mathbb{R}$ is upper semicontinuous.

(2) The map $\mathcal{M}^{\mathrm{Fr}\, K}(X)_w \ni \mu \longmapsto \mu(K) \in \mathbb{R}$ is continuous.

Lemma 2.1. *Let $\mu \in \mathcal{M}(M)$.*

The space $\mathcal{M}(M)_w$ admits a canonical contraction $\varphi_t(\nu) = (1-t)\nu + t\mu$ $(0 \le t \le 1)$.

Definition 2.2.

(1) A Radon measure $\mu \in \mathcal{M}(X)$ is said to be good if

(i) $\mu(p) = 0$ for any $p \in M$ and (ii) $\mu(U) > 0$ for any nonempty open subset U of X.

(2) $\mathcal{M}_g^A(X) = \{\mu \in \mathcal{M}^A(X) \mid \mu : \text{good}\} \quad (A \in \mathcal{B}(X))$.

Definition 2.3. Let $\mu \in \mathcal{M}_g^A(X)$.

(1) A Radon measure $\nu \in \mathcal{M}_g^A(X)$ is μ-biregular

if μ and ν have same null sets (i.e., $\mu(B) = 0$ iff $\nu(B) = 0$ for $B \in \mathcal{B}(X)$).

(2) $\mathcal{M}_g^A(X; \mu\text{-reg}) = \{\nu \in \mathcal{M}_g^A(X) \mid \nu(X) = \mu(X),\ \nu : \mu\text{-biregular}\}$.

2.2. **Homeomorphism groups.**

Let $\mathcal{H}(X)$ denote the group of homeomorphisms of X with the compact-open topology. It is known that $\mathcal{H}(X)$ is a separable completely metrizable topological group.

Definition 2.4. Let $A, C \subset X$.

(1) $\mathcal{H}_C(X, A) = \{h \in \mathcal{H}(X) \mid h|_C = id_C,\ h(A) = A\}$

(2) $\mathcal{H}_C^c(X, A) = \{h \in \mathcal{H}_C(X, A) \mid \operatorname{Supp} h : \text{compact}\}$

(3) For a subgroup G of $\mathcal{H}(X)$ let G_0 denote the connected component of id_M in G.

Let $\mu \in \mathcal{M}(X)$.

Definition 2.5. For $h \in \mathcal{H}(X)$ a measure $h_*\mu \in \mathcal{M}(X)$ is defined by $(h_*\mu)(B) = \mu(h^{-1}(B))$ $(B \in \mathcal{B}(X))$.

Definition 2.6. Let $h \in \mathcal{H}(X)$.

(1) h is μ-preserving if $h_*\mu = \mu$ (i.e., $\mu(h(B)) = \mu(B)$ for any $B \in \mathcal{B}(X)$).

(2) h is μ-biregular if h preserves μ-null sets (i.e., $\mu(h(B)) = 0$ iff $\mu(B) = 0$ for $B \in \mathcal{B}(X)$).

Definition 2.7.

(1) $\mathcal{H}_C(X, A;\ \mu\text{-reg}) = \{h \in \mathcal{H}_C(X, A) \mid h : \mu\text{-biregular}\}$

(2) $\mathcal{H}_C(X, A;\ \mu) = \{h \in \mathcal{H}_C(X, A) \mid h : \mu\text{-preserving}\}$

2.3. **Actions of homeomorphism groups on spaces of Radon measures.**

The topological group $\mathcal{H}(X, A)$ acts continuously on the space $\mathcal{M}_g^A(X)_w$ by $h \cdot \mu = h_*\mu$. For $\mu \in \mathcal{M}_g^A(X)$ the orbit map at μ is the map $\pi : \mathcal{H}(X, A) \to \mathcal{M}_g^A(X)_w : h \mapsto h_*\mu$. The group $\mathcal{H}(X, A; \mu)$ is the stabilizer of μ and it is the fiber of π at μ.

3. Compact manifold case

In this section we list topological properties of groups of measure-preserving homeomorphisms of compact n-manifolds. Suppose M is a compact connected n-manifold.

Theorem 3.1. **(Transitivity)** (von Neumann-Oxtoby-Ulam [7]) *Let* $\mu, \nu \in \mathcal{M}_g^\partial(M)$.
There exists $h \in \mathcal{H}_\partial(M)_0$ *with* $h_*\mu = \nu$ *iff* $\mu(M) = \nu(M)$.

Let $\mu \in \mathcal{M}_g^\partial(M)$. The topological group $\mathcal{H}(M; \mu\text{-reg})$ acts continuously on the space $\mathcal{M}_g^\partial(M; \mu\text{-reg})_w$ by $h \cdot \mu = h_*\mu$. The orbit map at μ is the map $\pi : \mathcal{H}(M; \mu\text{-reg}) \to \mathcal{M}_g^\partial(M; \mu\text{-reg})_w : h \mapsto h_*\mu$. The group $\mathcal{H}(M; \mu)$ is the stabilizer of μ and it is the fiber of π at μ.

Theorem 3.2. **(Sections of orbit maps)** (A. Fathi [5])
The orbit map $\pi : \mathcal{H}(M; \mu\text{-reg}) \to \mathcal{M}_g^\partial(M; \mu\text{-reg})_w$ *admits a continuous section* $\sigma : \mathcal{M}_g^\partial(M; \mu\text{-reg})_w \to \mathcal{H}_\partial(M; \mu\text{-reg})_0$.

Corollary 3.1.

(1) $\mathcal{H}(M; \mu\text{-reg}) \cong \mathcal{H}(M; \mu) \times \mathcal{M}_g^\partial(M; \mu\text{-reg})_w$.
(2) $\mathcal{H}(M; \mu)$ *is a strong deformation retract* (*SDR*) *of* $\mathcal{H}(M; \mu\text{-reg})$.

A. Fathi [5] also studied the properties of the inclusion $\mathcal{H}(M; \mu\text{-reg}) \subset \mathcal{H}(M)$.

Definition 3.1. A subset B of a space Y is homotopy dense (HD)
if there exists a homotopy $\varphi_t : Y \to Y$ such that $\varphi_0 = id_Y$ and $\varphi_t(Y) \subset B$ $(0 < t \leq 1)$.

Theorem 3.3. (A. Fathi [5])
$\mathcal{H}(M; \mu\text{-reg})$ *is "HD for maps from finite-dimensional spaces" in* $\mathcal{H}(M)$.
In particular, the inclusion $\mathcal{H}(M; \mu\text{-reg}) \subset \mathcal{H}(M)$ *is a weak homotopy equivalence.*

When M is a compact 2-manifold, the group $\mathcal{H}(M)$ is an ANR [6] and a ℓ_2-manifold cf.[4]. This implies the next consequences.

Corollary 3.2. *Suppose* M *is a compact connected 2-manifold.*

(1) (i) $\mathcal{H}(M; \mu\text{-reg})$ *is HD in* $\mathcal{H}(M)$. (ii) $\mathcal{H}(M; \mu\text{-reg})$ *is an ANR.*
(2) (i) $\mathcal{H}(M; \mu)$ *is a* ℓ_2*-manifold.* (ii) $\mathcal{H}(M; \mu)$ *is a SDR of* $\mathcal{H}(M)$.

The assertion (3) in Corollary 3.2 is deduced from the following characterization of ℓ_2-manifolds for topological groups.

Theorem 3.4. (T. Dobrowolski - H. Toruńczyk [4]) *Let* G *be a topological group.*
The space G *is a* ℓ_2*-manifold iff* G *is a separable, non-locally compact, completely metrizable ANR.*

4. Non-compact manifold case

R. Belranga [1, 2, 3] has extended some results in the previous setion to noncompact manifolds. To treat noncompact manifolds we have to include informations on ends of manifolds.

4.1. End compactification.

First we recall basic facts on end compactifications. Suppose X is a connected locally connected locally compact separable metrizable space. Let $\mathcal{C}(X)$ denote the set of connected components of X.

Definition 4.1.

(1) An end e of X is an assignment $e : \mathcal{K}(X) \ni K \longmapsto e(K) \in \mathcal{C}(X - K)$ such that $e(K_1) \supset e(K_2)$ for $K_1 \subset K_2$.

(2) $E(X)$ = the space of ends of X

(3) The end compactification of X is the space $\overline{X} = X \cup E(X)$ endowed with the topology prescribed by the following conditions :

  (i) X is an open subspace of $\overline{X}$.

  (ii) The fundamental open neighborhoods of $e \in E(X)$ are given by

$$N(e, K) = e(K) \cup \{e' \in E(X) \mid e'(K) = e(K)\} \quad (K \in \mathcal{K}(X)).$$

Let $\mu \in \mathcal{M}(X)$.

Definition 4.2.

(1) An end $e \in E(X)$ is said to be μ-finite if $\mu(e(K)) < \infty$ for some $K \in \mathcal{K}(X)$.

(2) $E_f(X; \mu) = \{e \in E(X) \mid e : \mu\text{-finite}\}$

Definition 4.3.

(1) $h \in \mathcal{H}(X)$ is μ-end-regular if h is μ-biregular and preserves the μ-finite ends of X.

(2) $\mathcal{H}_C(X; \mu\text{-end-reg}) = \{h \in \mathcal{H}_C(X) \mid h : \mu\text{-end-regular}\}$

4.2. Finite-end weak topology.

Let $\mu \in \mathcal{M}_g^A(X)$.

Definition 4.4.

(1) $\nu \in \mathcal{M}_g^A(X)$ is μ-end-biregular if
(i) ν is μ-biregular and (ii) $E_f(X, \nu) = E_f(X, \mu)$ (i.e., ν and μ have same finite ends).

(2) $\mathcal{M}_g^A(X; \mu\text{-end-reg}) = \{\nu \in \mathcal{M}_g^A(X) \mid \nu(X) = \mu(X),\ \nu : \mu\text{-end-biregular}\}$

Consider the subspaces $X \overset{i}{\subset} X \cup E_f(X; \mu) \subset X \cup E(X) = \overline{X}$. The inclusion $X \overset{i}{\subset} X \cup E_f(X; \mu)$ induces a natural injection

$$\begin{array}{rccc} i_* : & \mathcal{M}_g^A(X, \mu\text{-end-reg}) & \longrightarrow & \mathcal{M}_g^A(X \cup E_f(X; \mu))_w \\ & \nu & \longmapsto & i_*\nu \end{array}$$

Definition 4.5. The finite-end weak topology ew on $\mathcal{M}_g^A(X, \mu\text{-end-reg})$ is the topology induced by the injection $i_* : \mathcal{M}_g^A(X, \mu\text{-end-reg}) \to \mathcal{M}_g^A(X \cup E_f(X; \mu))_w$ (i.e., the weakest topology such that i_* is continuous).

Lemma 4.1. *The space $\mathcal{M}_g^A(X;\mu\text{-end-reg})_{ew}$ admits a contraction $\varphi_t(\nu) = (1-t)\nu + t\mu$ $(0 \le t \le 1)$.*

The topological group $\mathcal{H}(X, A; \mu\text{-end-reg})$ acts continuously on $\mathcal{M}_g^A(X;\mu\text{-end-reg})_{ew}$ by $h \cdot \nu = h_*\nu$. The orbit map at μ is the map $\pi : \mathcal{H}(X, A; \mu\text{-end-reg}) \to \mathcal{M}_g^A(X;\mu\text{-end-reg})_{ew} : h \mapsto h_*\mu$. The group $\mathcal{H}(X, A; \mu)$ is the stabilizer of μ and it is the fiber of π at μ.

4.3. Results of R. Berlanga.

Suppose M is a noncompact connected separable metrizable n-manifold and $\mu \in \mathcal{M}_g^\partial(M)$. R. Berlanga [1, 2, 3] obtained the following conclusions on the action of $\mathcal{H}(M;\mu\text{-end-reg})$ on $\mathcal{M}_g^\partial(M;\mu\text{-end-reg})_{ew}$.

Theorem 4.1. (Transitivity [1]) *Let $\mu, \nu \in \mathcal{M}_g^\partial(M)$.*
There exists $h \in \mathcal{H}_\partial(M)_0$ with $h_*\mu = \nu$ iff $\mu(M) = \nu(M)$ and $E_f(M,\mu) = E_f(M,\nu)$.

Theorem 4.2. (Sections of orbit maps [3])
The orbit map $\pi : \mathcal{H}(M;\mu\text{-end-reg}) \to \mathcal{M}_g^\partial(M;\mu\text{-end-reg})_{ew}$, $\pi(h) = h_*\mu$ admits a continuous section $\sigma : \mathcal{M}_g^\partial(M;\mu\text{-end-reg})_{ew} \to \mathcal{H}_\partial(M;\mu\text{-end-reg})_0$.

Corollary 4.1.
(1) *$\mathcal{H}_\partial(M;\mu\text{-end-reg}) \cong \mathcal{H}_\partial(M;\mu) \times \mathcal{M}_g^\partial(M;\mu\text{-end-reg})_{ew}$.*
(2) *$\mathcal{H}_\partial(M;\mu)$ is a SDR of $\mathcal{H}_\partial(M,\mu\text{-end-reg})$.*

At this moment we have no general results on the inclusion $\mathcal{H}_\partial(M,\mu\text{-end-reg}) \subset \mathcal{H}_\partial(M)$.

Problem 4.1. Is $\mathcal{H}_\partial(M,\mu\text{-end-reg})$ HD in $\mathcal{H}_\partial(M)$?

4.4. Homeomorphism groups of noncompact 2-manifolds.

We have made a comprehensive study on topological properties of groups of homeomorphisms of noncompact 2-manifolds [8, 9, 10, 11]. The results are summarized as follows.

Theorem 4.3. [9, 10] *Suppose M is a noncompact connected 2-manifold.*
(1) *$\mathcal{H}(M)_0$ is a ℓ_2-manifold.*
(2) $\mathcal{H}(M)_0 \simeq \begin{cases} \mathbb{S}^1 & (M = \mathbb{R}^2,\ \mathbb{S}^1 \times \mathbb{R},\ \mathbb{S}^1 \times [0,1),\ \textit{the open Möbius band}) \\ * & (\textit{otherwise}) \end{cases}$
(3) *$\mathcal{H}^{\mathrm{PL,c}}(M)_0$ is HD in $\mathcal{H}(M)_0$ (for any PL-structure on M).*

We also studied topological properties of the space of embeddings into 2-manifolds [8, 11]. Suppose M is a connected 2-manifold and X is a compact connected subpolyhedron of M. Let $\mathcal{E}(X, M)$ denote the space of embeddings of X into M with the compact-open topology and $\mathcal{E}(X, M)_0$ denote the connected component of the inclusion $i : X \subset M$ in $\mathcal{E}(X, M)$. We showed that $\mathcal{E}(X, M)$ is a ℓ_2-manifold and classified the homotopy type of $\mathcal{E}(X, M)_0$.

As an application of Theorem 4.3 we can give an affirmative answer to Problem 4.1 in dimension 2 and obtain some consequences on topological properties of groups of measure-preserving homeomorphisms of noncompact 2-manifolds [12].

Lemma 4.2. *Suppose M is a PL n-manifold and $\mu \in \mathcal{M}_g^\partial(M)$.*

There exists an isotopy φ_t of M such that $\varphi_0 = id_M$ and $\mathcal{H}^{\rm PL}(M) \subset \mathcal{H}(M; \mu\text{-end-reg})$ for the new PL - structure on M induced by φ_1 from the old PL - structure.

Corollary 4.2. *Suppose M is a noncompact connected 2-manifold.*

(1) (i) *$\mathcal{H}(M, \mu\text{-end-reg})_0$ is HD in $\mathcal{H}(M)_0$.* (ii) *$\mathcal{H}(M, \mu\text{-end-reg})_0$ is an ANR.*

(2) (i) *$\mathcal{H}(M; \mu)_0$ is a ℓ_2-manifold.* (ii) *$\mathcal{H}(M; \mu)_0$ is a SDR of $\mathcal{H}(M)_0$.*

REFERENCES

[1] R. Berlanga and D. B. A. Epstein, Measures on sigma-compact manifolds and their equivalence under homeomorphism, *J. London Math. Soc. (2)* 27 (1983) 63 - 74.

[2] R. Berlanga, A mapping theorem for topological sigma-compact manifolds, *Compositio Math.*, 63 (1987) 209 - 216.

[3] R. Berlanga, Groups of measure-preserving homeomorphisms as deformation retracts, *J. London Math. Soc. (2)* 68 (2003) 241 - 254.

[4] T. Dobrowolski and H. Toruńczyk, Separable complete ANR's admitting a group structure are Hilbert manifolds, *Top. Appl.*, 12 (1981) 229 - 235.

[5] A. Fathi, Structures of the group of homeomorphisms preserving a good measure on a compact manifold, *Ann. scient. Éc. Norm. Sup. (4)* 13 (1980) 45 - 93.

[6] R. Luke and W. K. Mason, The space of homeomorphisms on a compact two - manifold is an absolute neighborhood retract, *Trans. Amer. Math. Soc.*, 164 (1972), 275 - 285.

[7] J. Oxtoby, and S. Ulam, Measure preserving homeomorphisms and metrical transitivity, *Ann. of Math.*, 42 (1941) 874 - 920.

[8] T. Yagasaki, Spaces of embeddings of compact polyhedra into 2-manifolds, *Topology Appl.*, 108 (2000) 107 - 122.

[9] T. Yagasaki, Homotopy types of homeomorphism groups of noncompact 2-manifolds, *Topology Appl.*, 108 (2000) 123 - 136.

[10] T. Yagasaki, The groups of PL and Lipschitz homeomorphisms of noncompact 2-manifolds, *Bulletin of the Polish Academy of Sciences, Mathematics*, 51(4) (2003), 445 - 466.

[11] T. Yagasaki, Homotopy types of the components of spaces of embeddings of compact polyhedra into 2-manifolds, *Topology Appl.*, 153 (2005) 174 - 207.

[12] T. Yagasaki, Groups of measure-preserving homeomorphisms of noncompact 2-manifolds, preprint, arXiv math.GT/0507328.